

hep-th/0407179  
 BRX-TH-545  
 BOW-PH-132

# MANY ROADS LEAD TO $\mathcal{N} = 2$ SEIBERG–WITTEN THEORY\*

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The Seiberg–Witten solution plays a central role in the study of  $\mathcal{N} = 2$  supersymmetric gauge theories. As such, it provides a proving ground for a wide variety of techniques to treat such problems. In this review we concentrate on the role of IIA string theory/M theory and the Dijkgraaf–Vafa matrix model, though integrable models and microscopic instanton calculations are also of considerable importance in this subject.

## Outline

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## 1. Introduction

Neither of the authors has had the privilege of collaborating with Stanley Deser, much to our regret. Since Stanley is still in the full bloom of his career, there is still a great deal of time to remedy this. Stanley has been a wonderful colleague, with whom we have enjoyed talking about physics

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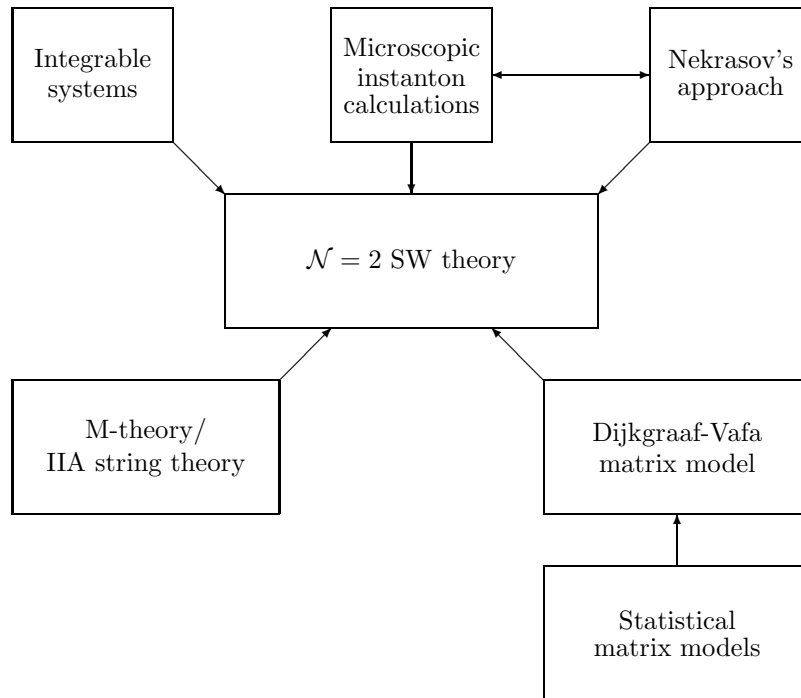
\*Based on the talk by H. Schnitzer at the Deserfest, University of Michigan, 3–5 April, 2004

<sup>†</sup>Research supported in part by the NSF under grant PHY-0140281.

<sup>‡</sup>Research supported in part by the DOE under grant DE-FG02-92ER40706.

and many other topics for many, many years. We hope this situation will continue well into the future. We dedicate this article to a celebration of Stanley's achievements.

The theme of our discussion is to argue that  $\mathcal{N} = 2$  Seiberg–Witten (SW) theory<sup>1</sup> lies at the cross-roads of many different aspects of gauge theories. The various paths that intersect SW theory are quantum field theory, string theory, integrable models, and most recently the Dijkgraaf–Vafa (DV) matrix model<sup>2</sup>. Because of time limitations, we will not survey all these roads, but concentrate on just two because 1) they are of current interest and 2) they seem to involve drastically different concepts, which nevertheless lead to a better understanding of SW theory. These different approaches to a common problem enhance our confidence in the various methods, and lends support to applying these techniques to new problems.



**Figure 1: Roads to Seiberg-Witten theory**

In our own endeavors on SW theory we have been fortunate to have had a number of excellent collaborators: Isabel Ennes, Marta Gómez-Reino, Carlos Lozano, Henric Rhedin, and Niclas Wyllard.

## 2. Review of $\mathcal{N} = 2$ Seiberg–Witten theory

$\mathcal{N} = 2$  supersymmetric (susy) Yang–Mills theory with gauge group  $\mathcal{G}$  and hypermultiplets in representation  $R$  can be described in the low-energy region exactly by the SW effective theory. The underlying microscopic Lagrangian is

$$\begin{aligned} \mathcal{L}_{\text{micro}} = & -\frac{1}{4g^2} F_{\mu\nu}^a F^{\mu\nu a} + \frac{\theta}{32\pi^2} F_{\mu\nu}^a \tilde{F}^{\mu\nu a} \\ & + \frac{1}{g^2} \text{tr} D_\mu \phi D^\mu \phi^\dagger - \frac{1}{2g^2} \text{tr} [\phi, \phi^\dagger]^2 \end{aligned} \quad (2.1)$$

plus fermion and hypermultiplet terms. In eq. (2.1),  $F_{\mu\nu}^a$  and  $\phi^a$  denote the gauge field strength and the bosonic components of the  $\mathcal{N} = 1$  chiral superfield belonging to the  $\mathcal{N} = 2$  vector multiplet respectively, both in the adjoint representation. The vacuum is defined by

$$[\phi, \phi^\dagger] = 0 \quad (2.2)$$

which implies that  $\phi$  may be diagonalized to give

$$\phi = \text{diag} (a_i) . \quad (2.3)$$

Generically this breaks  $\mathcal{G}$  to  $U(1)^{\text{rank } \mathcal{G}}$ . If only  $\phi$  acquires a vacuum expectation value (vev), this defines the Coulomb branch. If only the scalar fields in the matter hypermultiplet have vevs, this defines the Higgs branch. We focus on the Coulomb branch.

The Seiberg–Witten program for  $\mathcal{N} = 2$  susy gauge theories centers on the low-energy effective Lagrangian<sup>1</sup>

$$\mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \text{Im} \left[ \int d^4\theta \frac{\partial \mathcal{F}(A)}{\partial A^i} \bar{A}^i + \frac{1}{2} \int d^2\theta \frac{\partial^2 \mathcal{F}(A)}{\partial A^i \partial A^j} W^i W^j \right] \quad (2.4)$$

where  $\mathcal{F}(A)$  is the holomorphic prepotential and  $A^i$  are  $\mathcal{N} = 1$  chiral superfields. On the Coulomb branch one may write

$$\begin{aligned} \mathcal{L}_{\text{eff}} = \frac{1}{4\pi} \left[ -\frac{1}{4} \text{Im}(\tau_{ij}) F_{\mu\nu}^i F^{\mu\nu j} + \frac{1}{4} \text{Re}(\tau_{ij}) F_{\mu\nu}^i \tilde{F}^{\mu\nu j} \right. \\ \left. + \text{Im} \partial_\mu (a^\dagger)^j \partial^\mu (a_D)_j + \text{fermions} \right] \end{aligned} \quad (2.5)$$

where

$$a_i = \text{order parameters} \quad (2.6a)$$

$$(a_D)_j = \frac{\partial \mathcal{F}(a)}{\partial a_j} = \text{dual order parameters} \quad (2.6b)$$

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}(a)}{\partial a_i \partial a_j} = \text{coupling matrix or period matrix.} \quad (2.6c)$$

One requires  $\text{Im } \tau_{ij} \geq 0$  for a positive kinetic energy.

As a result of non-renormalization theorems, the holomorphic prepotential only receives perturbative corrections at one-loop, but there is an infinite series of instanton contributions. Thus

$$\mathcal{F}(A) = \mathcal{F}_{\text{classical}} + \mathcal{F}_{1\text{-loop}} + \sum_{d=1}^{\infty} \Lambda^{bd} \mathcal{F}_{d\text{-instanton}} \quad (2.7)$$

where

$$\begin{aligned} \Lambda &= \text{the quantum scale (Wilson cutoff),} \\ b &= \text{the coefficient of the } \beta \text{ function,} \end{aligned} \quad (2.8)$$

with  $b$  positive for asymptotically-free theories. For example, for  $\mathcal{N} = 2$   $\text{SU}(N)$  theory with  $N_f$  hypermultiplets in the fundamental representation

$$\begin{aligned} \mathcal{F}_{1\text{-loop}}(a) &= \frac{i}{8\pi} \sum_{i=1}^N \sum_{j \neq i}^N (a_i - a_j)^2 \log \left( \frac{a_i - a_j}{\Lambda} \right)^2 \\ &\quad - \frac{i}{8\pi} \sum_{i=1}^N \sum_{I=1}^{N_f} (a_i + m_I)^2 \log \left( \frac{a_i + m_I}{\Lambda} \right)^2 \end{aligned} \quad (2.9)$$

yielding

$$\tau_{ij}(a) \sim \log(a_i - a_j) + \dots \quad (i \neq j) \quad (2.10)$$

at large  $a$ . Significantly  $\tau_{ij}$  is not single-valued, and a Riemann surface emerges for which  $\tau_{ij}$  is the period matrix.

In general, the Seiberg–Witten data needed to find the prepotential are:

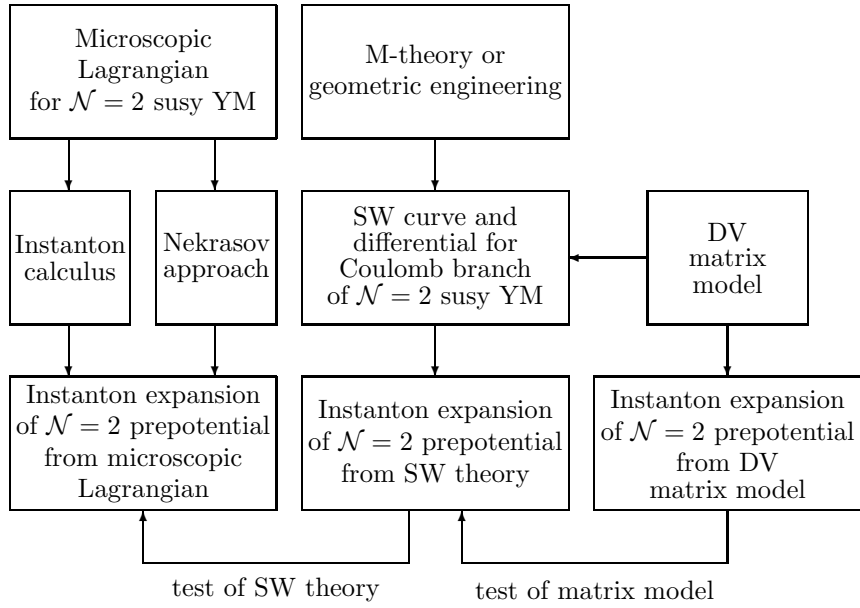
- 1) A Riemann surface or algebraic curve specific to the gauge group and matter content, dependent on moduli  $u_i$ , which are related to the order parameters  $a_i$  ;
- 2) a preferred meromorphic 1-form  $\lambda$  = the SW differential;
- 3) a canonical basis of homology cycles on the surface  $(A_k, B_k)$ .

Given this data, the program to solve for the prepotential is:

- 1) Compute the period integrals of the SW differential

$$2\pi i a_k = \oint_{A_k} \lambda, \quad 2\pi i (a_D)_k = \oint_{B_k} \lambda. \quad (2.11)$$

- 2) Integrate  $a_D$  to find  $\mathcal{F}(a)$ .
- 3) Test this against explicit results from  $\mathcal{L}_{\text{micro}}$  where possible.
- 4) Test this against the predictions of the DV matrix model.

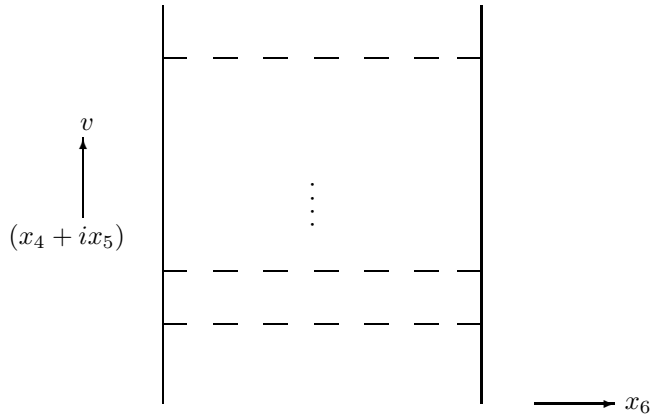


**Figure 2: Computing the instanton expansion**

### 3. The SW curve from IIA string theory/M theory

Klemm, *et. al.*,<sup>3</sup> demonstrated that SW theory could be derived from string theory, using a technique called geometric engineering. Subsequently, Witten<sup>4</sup> gave a systematic method to find SW curves by lifting 10-dimensional IIA string theory to 11-d M-theory. In IIA language, this

involve configurations of two (or more) parallel NS5 branes spanned by a number of D4 branes between each NS5 neighboring pair. For pure  $\mathcal{N} = 2$   $SU(N)$  gauge theory (with no hypermultiplets), one has two parallel NS5 branes, connected by  $N$  D4 branes, as shown in fig. 3.



**Figure 3: IIA brane configuration for  $\mathcal{N} = 2$   $SU(N)$  gauge theory**

In the lift to 11-d one can visualize the IIA brane picture as a Riemann surface, with (fattened) lines for the D4 branes becoming branch-cuts which connect two Riemann sheets, *i.e.*, the two NS5 branes. This gives the hyperelliptic curve

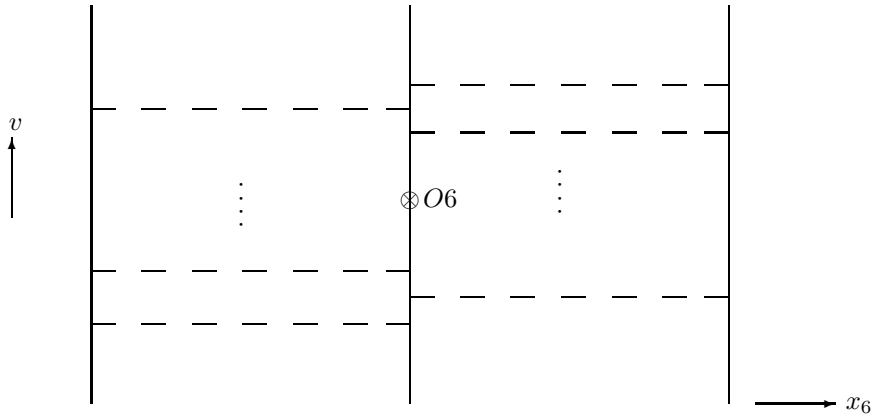
$$y^2 = \prod_{i=1}^N (x - e_i)^2 - 4\Lambda^{2N}. \quad (3.1)$$

More interesting are the theories yielding *non*-hyperelliptic curves, particularly since, for a long time, string theory was the only method available to obtain such curves. For example, the curve for  $SU(N_1) \times SU(N_2)$  gauge theory, with hypermultiplets in bifundamental representations  $(N_1, \bar{N}_2)$  and  $(\bar{N}_1, N_2)$ , is obtained from 3 parallel NS5 branes, with  $N_1$  D4-branes between the left and middle NS5 branes, and  $N_2$  D4-branes between the middle and right NS5's. Upon lift to M-theory, this configuration yields a 3-sheeted surface of the form<sup>3,4</sup>

$$y^3 + A(x)y^2 + B(x)y + C(x) = 0 \quad (3.2)$$

where the polynomials  $A(x)$ ,  $B(x)$ , and  $C(x)$  depend on the details of the

theory. Even more interesting is the case when an orientifold  $O6$  plane is placed on the middle NS5 brane<sup>5</sup>, as shown in fig. 4. The presence of the orientifold plane implies a reflection symmetry on the brane-picture, and an involution imposed on a curve of type (3.2). An  $O6^+$  plane corresponds to  $SU(N) + \square\square$  gauge theory, while an  $O6^-$  plane yields  $SU(N) + \square$ .



**Figure 4: IIA brane configuration for  $\mathcal{N} = 2$   $SU(N) + \square\square/\square$**

One of the significant contributions of the Brandeis group is the development of methods for finding the instanton expansion for non-hyperelliptic curves such as (3.2), and others<sup>6</sup>. In order to compute the instanton expansion for the prepotential one needs to compute

$$a_i \sim \oint_{A_i} \frac{x dy}{y}, \quad (a_D)_i \sim \oint_{B_i} \frac{x dy}{y} \quad (3.3)$$

in some approximation scheme since an exact solution seems inaccessible. The method developed by our group was hyperelliptic perturbation theory<sup>6</sup>, *i.e.*, a systematic expansion of the non-hyperelliptic curve about a fiducial hyperelliptic curve.

The treatment of  $SU(N) + \square\square$  and  $SU(N) + \square$  gauge theories typify the method, but a number of other examples including other groups were also considered. Explicit calculations enabled us to obtain  $\mathcal{F}_{1-\text{inst}}$  from our methods. More recent calculations<sup>7,8</sup> involving the renormalization group (RG) have simplified the method, and allow an easier access to two or more instanton contributions. In *every* case, the results from our hyperelliptic

expansion agree with those of microscopic calculations, when available. The most systematic of these are by Nekrasov and others<sup>9</sup>, and recently by Mariño and Wyllard<sup>10</sup>. This agreement supports the M-theory approach to SW theory, as well as the validity of our approximation methods.

#### 4. The Dijkgraaf–Vafa matrix model approach

A major advance in our understanding of susy gauge theory was provided by Dijkgraaf, Vafa, and collaborators<sup>2</sup>. They showed that a suitably formulated matrix model will describe the low-energy physics of  $\mathcal{N} = 1$  or  $\mathcal{N} = 2$  susy gauge theories. This is proved by showing that, in the computation of the effective superpotential  $W_{\text{eff}}$  and the period matrix  $\tau_{ij}$  of the gauge theory, the space-time part of Feynman integrals cancels, leaving a zero-dimensional theory, *i.e.*, a matrix model<sup>11</sup>. An alternate proof<sup>12</sup> shows that the generalized Konishi anomaly equations of the gauge theory have a form identical to those of the resolvent operator of the matrix model.

We will describe the application of the DV model to SW theory. To be specific, consider the matrix model appropriate to  $\mathcal{N} = 2$   $U(N)$  theory with  $N_f$  hypermultiplets in the fundamental ( $\square$ ) representation. Consider  $M \times M$  matrices  $\Phi$  together with  $M$ -vectors  $Q^I$ ,  $\tilde{Q}_I$  ( $I = 1$  to  $N_f$ ). One wishes to compute the partition function<sup>2</sup>

$$Z = \frac{1}{\text{vol } G} \int d\Phi dQ^I d\tilde{Q}_I \exp \left[ -\frac{1}{g_s} W(\Phi, Q, \tilde{Q}) \right] \quad (4.1)$$

with  $G$  the unbroken matrix gauge group. The choice of  $W(\Phi, Q, \tilde{Q})$  determines the physics. For our example

$$W = W_0(\Phi) + W_{\text{matter}}(\Phi, Q, \tilde{Q}) \quad (4.2)$$

where

$$W'_0(x) = \alpha \prod_{i=1}^N (x - e_i) \quad (4.3)$$

breaks  $\mathcal{N} = 2$  susy to  $\mathcal{N} = 1$ . One takes  $\alpha \rightarrow 0$  at the end of the calculation to recover  $\mathcal{N} = 2$  physics from the  $\alpha$ -independent quantities  $W_{\text{eff}}$  and  $\tau_{ij}$ . The matter interaction in (4.2) is taken to be of the same form as the analogous superpotential in the underlying gauge theory. For the theory we are considering,<sup>13,14</sup>

$$W_{\text{matter}} = \sum_{a,b}^M \sum_{I=1}^{N_f} \tilde{Q}_{aI} \Phi^a_b Q^{bI}. \quad (4.4)$$

To evaluate the partition function (4.1), one expands the action about a stationary point<sup>2</sup>

$$\Phi = \Phi_0, \quad Q = \tilde{Q} = 0 \quad (4.5)$$

with fluctuations

$$\Phi = \Phi_0 + \Psi = \text{diag}(e_i \mathbb{1}_{M_i}) + (\Psi_{ij}) \quad (4.6)$$

where  $\Psi_{ij}$  is an  $M_i \times M_j$  matrix, and

$$\sum_{i=1}^N M_i = M, \quad G = \prod_{i=1}^N U(M_i). \quad (4.7)$$

One makes the gauge choice  $\Psi_{ij} = 0$  for  $i \neq j$ , which requires the introduction of ghost matrix fields<sup>18</sup>. One obtains the Feynman rules from the expansion of the action, as well as for the ghost contributions, from which one can do a perturbative evaluation of the partition function.

In the limit  $M_i \gg 1$  for all  $i$ , with

$$S_i \equiv g_s M_i = \text{finite}, \quad (4.8)$$

one may express the partition integral in a topological expansion<sup>2</sup>,

$$Z = \exp F(e, S) = \exp \sum_{\chi} \frac{1}{g_s^\chi} F_\chi(e, S) \quad (4.9)$$

where  $F(e, S)$  is the free-energy and  $\chi = 2 - 2g - h$  is the Euler number for a two-dimensional surface with  $g$  handles and  $h$  holes. The leading terms are

$$F(e, S) = \frac{1}{g_s^2} F_{\text{sphere}}(e, S) + \frac{1}{g_s} F_{\text{disk}}(e, S) + \dots \quad (4.10)$$

where the sphere contribution has  $g = h = 0$ , while the disk contribution has  $g = 0$ ,  $h = 1$ , the latter arising from diagrams with  $Q$  or  $\tilde{Q}$  running along the boundary. (The  $h \geq 2$  diagrams are suppressed relative to  $\chi = 1$  for the limit  $g_s \rightarrow 0$ ,  $M_i \rightarrow \infty$ , with  $g_s M_i = S_i$  fixed.)

On the gauge theory side, the gauge symmetry is broken to

$$U(N) \rightarrow \prod_i U(N_i) \quad \sum_i N_i = N \quad (4.11)$$

It is important to emphasize that  $M_i \rightarrow \infty$  on the matrix-model side, while  $N_i$  remains finite on the gauge theory side. In order to study a generic point on the Coulomb branch of the  $\mathcal{N} = 2$  SW theory, one takes  $N_i = 1$  for all  $i$ , that is,

$$U(N) \rightarrow [U(1)]^N. \quad (4.12)$$

The effective superpotential of the gauge theory is then given by<sup>2,13,14</sup>

$$W_{\text{eff}}(e, S) = - \sum_{i=1}^N \frac{\partial F_{\text{sphere}}(e, S)}{\partial S_i} - F_{\text{disk}}(e, S) + 2\pi i \tau_0 \sum_{i=1}^N S_i \quad (4.13)$$

where  $\tau_0 = \tau(\Lambda_0)$  is the gauge coupling of  $U(N)$  at scale  $\Lambda_0$ . To proceed, one finds the extremum

$$\left. \frac{\partial W_{\text{eff}}}{\partial S_i} \right|_{\langle S_j \rangle} = 0 \quad (4.14)$$

which defines  $\langle S_i \rangle$ .

The period matrix is then obtained from *only* the sphere contribution to the free-energy

$$\tau_{ij}(e) = \frac{1}{2\pi i} \left. \frac{\partial^2 F_{\text{sphere}}(e, S)}{\partial S_i \partial S_j} \right|_{\langle S_i \rangle} \quad (4.15)$$

One lets  $\alpha \rightarrow 0$  to obtain  $\mathcal{N} = 2$  results. (Actually, however, eq. (4.15) is independent of  $\alpha$ .)

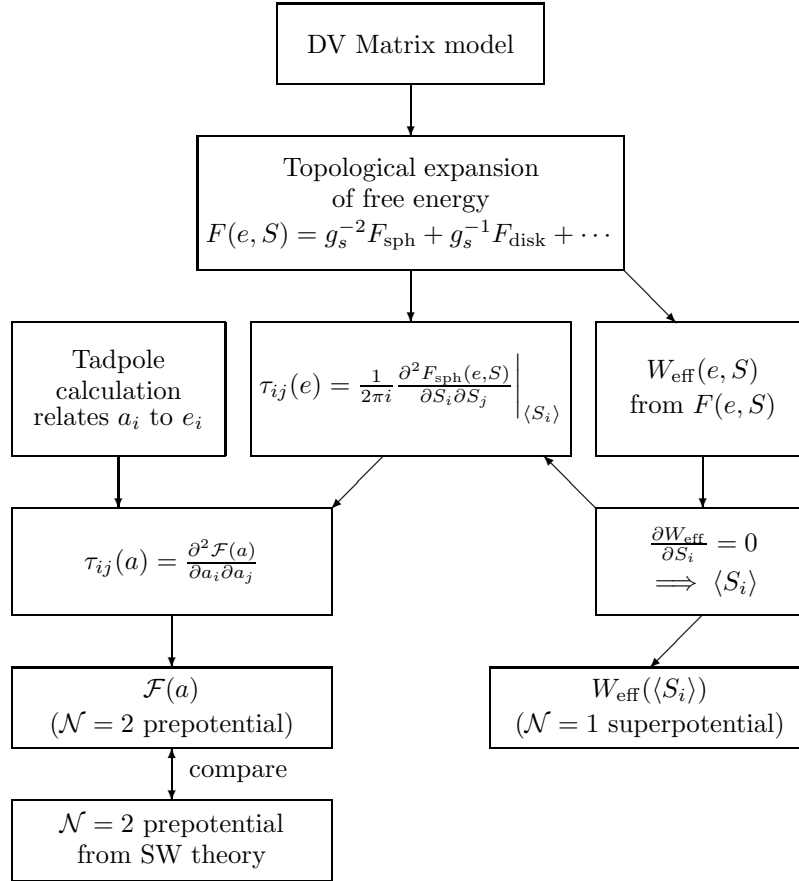
While eq. (4.15) gives the matrix of  $U(1)^N$  couplings as a function of the  $\{e_i\}$ , the period matrix (2.6c) from SW theory,

$$\tau_{ij}(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial a_i \partial a_j}, \quad (4.16)$$

is expressed in terms of *different* parameters on moduli space. The  $\{a_i\}$  are physical order parameters, while  $\{e_i\}$  are “bare” order parameters, so one needs a relation between  $a_i$  and  $e_i$ . One can show that the desired relation is<sup>13</sup>

$$a_i = e_i + \left[ \sum_{j=1}^N \frac{\partial}{\partial S_j} g_s \langle \text{tr } \Psi_{ii} \rangle_{\text{sphere}} + \langle \text{tr } \Psi_{ii} \rangle_{\text{disk}} \right]_{\langle S \rangle} \quad (4.17)$$

where  $\text{tr}$  is the trace over the  $i^{\text{th}}$  diagonal block in  $\Psi$  only. The computation of (4.17) involves the calculation of tadpole diagrams with external  $\Psi_{ii}$  legs in the matrix model. Using eq. (4.17), one may re-express (4.15) as a function of  $a_i$ . Finally, integration of  $\tau_{ij}(a)$  yields the  $\mathcal{N} = 2$  prepotential  $\mathcal{F}(a)$ . The procedure just outlined for calculating  $\mathcal{F}(a)$  is shown schematically in fig. 5.



**Figure 5: Matrix model flow chart**

In practice, one computes the matrix-model quantities to a certain power in  $\Lambda$ , which increases with the number of loops in the matrix model, but corresponds to the instanton expansion in gauge theory. That is, non-perturbative information in the gauge theory is obtained from perturbative calculations in the matrix model! In these calculations  $(n+1)$ -loop perturbation theory in the matrix model corresponds to the  $n$ -instanton term of the prepotential of the gauge theory<sup>13</sup>. Recent work<sup>8</sup> using the renormalization group has improved the situation, so that only  $n$ -loops in the matrix

model theory are required to obtain  $n$ -instanton accuracy in the gauge theory. Another improvement<sup>15</sup> allows one to obtain the one-instanton prepotential for  $U(N) + N_f \square$  using only the contribution from  $F_{\text{sphere}}$ . In *every* case, our matrix model calculations agree with results from “conventional” SW theory, and microscopic calculations.

From the matrix model approach, one can also directly derive<sup>18,13</sup> the SW curve and differential from matrix-model resolvent equations. Particularly noteworthy is the treatment of *non*-hyperelliptic curves within the context of matrix models<sup>16</sup>. It is possible to derive the correct SW curve for  $U(N) + \square\square$  and  $U(N) + \begin{smallmatrix} \square & \\ & \square \end{smallmatrix}$  gauge theories, as well as the one-instanton contribution to the prepotential. There are subtle points<sup>16</sup> in constructing  $\tau_{ij}$  for these two theories, and additional subtle issues<sup>16,17</sup> in choosing the correct matrix-model vacuum state for  $U(N) + \begin{smallmatrix} \square & \\ & \square \end{smallmatrix}$ . When these issues are dealt with correctly, one obtains agreement with our previous results from hyperelliptic perturbation theory, as well as microscopic calculations of Nekrosov and others,<sup>9</sup> and Mariño and Wyllard<sup>10</sup>.

## 5. Concluding remarks

Very often a single problem plays a central role in testing a wide variety of methods. Seiberg–Witten theory may not be the hydrogen atom of strongly interacting gauge theories, but it does seem to have a privileged position in regard to  $\mathcal{N} = 2$  susy gauge theories. As our initial road-map indicated, SW theory can be treated by rather diverse techniques, among them IIA string theory/M theory, the DV matrix model, microscopic instanton calculations, and integrable systems<sup>19,20</sup>. We have surveyed just two of these approaches due to time limitations, but the subject is vast and still developing in the other areas as well.

HJS wishes to thank the organizers of the Deserfest for the opportunity to honor Stanley with this survey of topics which have occupied us for a long time.

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